Interaction of N atoms with the radiation field in the restricted rotating-wave approximation. II. Numerical results

1979 J. Phys. A: Math. Gen. 12 2225

(http://iopscience.iop.org/0305-4470/12/11/030)

J. Phys. A: Math. Gen., Vol. 12, No. 11, 1979. Printed in Great Britain

Interaction of N atoms with the radiation field in the restricted rotating-wave approximation: II. Numerical results

Miguel Orszag

Department of Physics, Ryerson Polytechnical Institute, Toronto, Ontario, Canada

Received 16 August 1978, in final form 23 November 1978

Abstract. A numerical analysis of the solution of the problem of the interaction of N atoms via the electromagnetic field is presented, using the analytical results of paper I (Orszag 1979). Several initial conditions are considered for the cases of both spontaneous and stimulated emission. Results show that there is an excellent agreement between the present work and previous numerical results (Walls and Barakat 1970) in the super-radiant case, while there are differences when the system is initially prepared in the state of complete inversion. These differences are attributed to the restricted rotating-wave approximation, which is the starting assumption in the present work.

1. Introduction

In paper I (Orszag 1979) we presented the closed analytical solution to the fundamental problem of the interaction between N two-level systems and electromagnetic radiation in the resonance case, assuming the restricted rotating-wave approximation (in the terminology of paper I) for the model.

Consider the Dicke Hamiltonian:

$$H/\hbar K = \omega_1 \hat{N} + R_3 \Delta_1 + (a + a^+)(R^+ + R^-) \tag{1}$$

where

$$\hat{N} = a^+ a + R_3, \qquad \omega_1 = \omega/K, \qquad \Delta_1 = (\omega_0 - \omega)/K \tag{2}$$

and K is the coupling constant between the atoms and the field.

Equation (1) is not exactly Dicke's model (Dicke 1954) since we have not included the rotating-wave approximation which is present in every previous work on the subject. Instead we assumed in paper I that

$$[\hat{N}, H] = 0 \tag{3}$$

which is the restricted rotating-wave approximation.

Using the unscrambling theorem of Arecchi *et al* (1972), we proved that the n-photon probability at the scaled time τ is given by

$$\begin{aligned} |p(n,\tau)|^2 = {} & (n!)^{-3} 2^{-4r} \left[\prod_{i=1}^{n} (2r+1-i)(i) \right] \\ & \times \left(\sum_{s_1=0}^{n} \sum_{s_2=0}^{2r-n} \binom{n}{s_1} \binom{2r-n}{s_2} (-1)^{n-s_1} [2\tau(s_1+s_2-r)]^n \right. \\ & \left. \times \exp(-2\tau^2)(s_1+s_2-r)^2 \right)^2 \left(\sum_{n=0}^{n_{\max}} |p(n,\tau)|^2 \right)^{-1}, \end{aligned} \tag{4}$$

for the case when the system was initially prepared in the state of complete inversion.

The general result, for both spontaneous and stimulated emission ($n(0) \neq 0$), and when the system was prepared in any initial state, is the following:

$$
\begin{aligned}
|p(n,\tau)|^2 = \Bigg[\sum_{q=0}^{\text{upper}\, q} \Bigg\{ \Bigg[\prod_{s=1}^{q} (r-m(0)-s+1)(r+m(0)+s) \Bigg]^{1/2} \\
\times \Bigg[\prod_{t=1}^{q+\Delta n} (r+m(0)+q-t+1)(r-m(0)-q+t) \Bigg]^{1/2} (-\mathrm{i}\tau)^{2q+\Delta n} \\
\times \sum_{v=0}^{\infty} \Bigg\{ \Bigg[\prod_{p=1}^{v} \sum_{\omega=0}^{v} (c'_p)^{\omega}/\omega! \Bigg]_{\Sigma p\omega = v} \Bigg(\sum_{j=0}^{\text{upper}\, j} [(2(q+v)+\Delta n)! \\
\times (n(0)!n!)^{1/2}]/[2^{q+v-j}(q+v-j)!(n(0)-j)!j!(j+\Delta n)!] \Bigg) \Bigg\} \\
\times (q!(q+\Delta n)!)^{-1} \Bigg\} \Bigg]^2 \Bigg(\sum_{n=0}^{n_{\max}} |p(n,\tau)|^2 \Bigg)^{-1}. \qquad (5)
\end{aligned}
$$

where

$$a_p = (-1)^{p-1} 2^{2p} (2^{2p-1} - 1) B_{2p} \tau^{2p} / (p)(2p)!,$$

$$b_p = (-1)^{p} 2^{2p-1} (2^{2p} - 1) B_{2p} \tau^{2p} / (p)(2p)!,$$

$$c'_p = (2q + \Delta n) a_p + 2(m(0)+q) b_p,$$

and the Bernoulli numbers are given by

$$B_{2p} = (-1)^{p-1} \frac{(2)(2p)!}{(2\pi)^{2p}} \sum_{K=1}^{\infty} \frac{1}{K^{2p}}. \qquad (6)$$

$\Delta n = n - n(0), \qquad M = n(0) + m(0) = n + m,$

$$\text{upper}\, q = \begin{cases} n(0) & \text{if } M \leq r \\ n(0) + r - M & \text{if } M > r, \end{cases}$$

$$\text{upper}\, J = \begin{cases} n(0) & \text{if } n(0) < q + v \\ q + v & \text{if } n(0) \geq q + v. \end{cases}$$

When equation (5) is used for $n(0) = 0$, a general expression for the n-photon probability is obtained in the case of spontaneous emission.

An APL computer program was generated to study the numerical results for different initial configurations, in both spontaneous and stimulated emission, using equation (5). Another smaller program (FORTRAN) was used for the case of complete inversion ($n(0) =$, $M = r$, equation (4)). This paper deals with these results.

Section 2 is devoted to studying the spontaneous emission from N atoms prepared initially in the following states:

2.1. Super-radiant ($M = 0$)

2.2. State of complete inversion ($M = r$)

2.3. An intermediate state between the above two cases.

This section is concluded with the comparison of our results with previous work.

In § 3, we give results for stimulated emission from N atoms, for several values of the parameters M, r, $n(0)$ and τ.

Finally, a discussion of the results is given in § 4.

2. Spontaneous emission from N atoms

2.1. *System initially prepared in the super-radiant state* ($M = 0$)

The APL computer program was run using equation (5) for the cases $M = 0$, $r = 24$ and $M = 0$, $r = 50$. In both cases the variable used was n, τ being a parameter for each curve with value ranging from 0·016 to 0·2 in the first case and from 0·016 to 0·05 in the second case (figures 1 and 2 respectively).

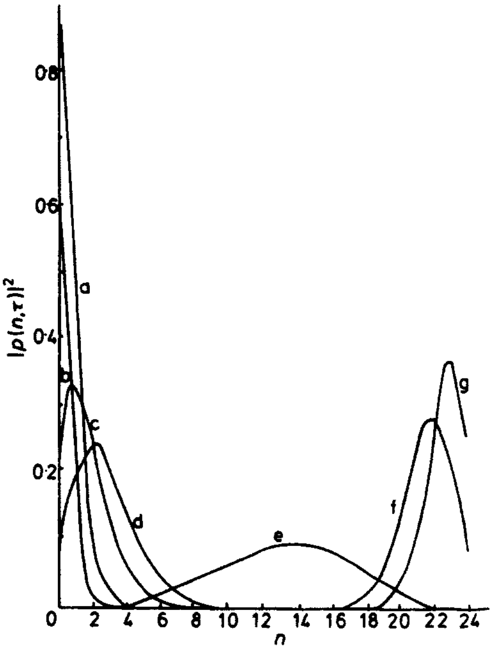

Figure 1. n-photon probability distribution versus the number of photons n for spontaneous emission ($n(0) = 0$), when the atoms are initially prepared in the super-radiant state ($M = 0$) and $r = 24$. $\tau = 0{\cdot}016$ (a), 0·032 (b), 0·05 (c), 0·065 (d), 0·1 (e), 0·15 (f) and 0·2 (g).

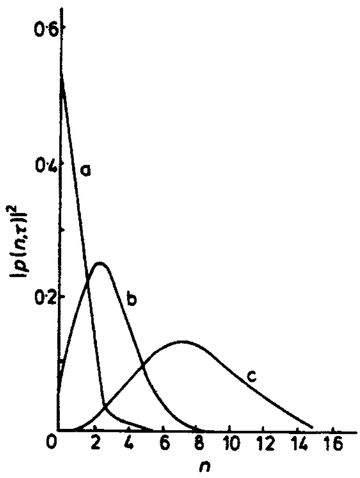

Figure 2. As figure 1 with $r = 50$. $\tau = 0{\cdot}016$ (a), 0·032 (b) and 0·05 (c).

Figure 1 compares very closely with the numerical work of Walls and Barakat (1970). They found a numerical solution of Dicke's model for the case $r = 24$, $M = 0$ (in our notation), as applied to a parametric amplifier. Their curves agree very well with figure 1.

According to an approximate scheme of Bonifacio and Preparata (1970) the n-photon probability for the atoms initially prepared in the super-radiant state is given by

$$|p(n, \tau)|^2 = \exp(-r^2\tau^2)(r\tau)^{2n}/n!. \tag{7}$$

Equation (7) is a good approximation, provided that $\tau \ll r^{-1/2}$. For the case $r = 24$, this approximation should be a good one if $\tau \ll 0{\cdot}204$. In table 1 we compare our results with the ones obtained from equation (7) for $\tau = 0{\cdot}016$, $r = 24$ and $M = 0$.

Table 1. Comparison of the n-photon probability of Bonifacio *et al* with our results, in the case $M = 0$, $r = 24$ and $\tau = 0{\cdot}016$, for $n = 0, 1, 2$.

	$\|p(n, \tau)\|^2$	
n	Exact solution	Bonifacio *et al*
0	0·857 58	0·862 96
1	0·131 79	0·127 24
2	0·000 51	0·000 46

2.2. System initially prepared in the complete inversion state ($M = r$).

The FORTRAN program was run using equation (4) for $M = r = 12$. The values of the parameter τ were 0·1 (figure 3), 0·3 (figure 4) and 0·5 (figure 5).

These curves, although qualitatively showing oscillatory behaviour similar to Walls' and Barakat's results, do not agree numerically. What we observe in the present solution is a systematic deviation from previous work, in the case when the system has been prepared initially in the state of complete inversion. In paper I it is also shown that for the case $M = r = 0{\cdot}5$ our solution does not agree with the results of Jaynes and Cummings (1963). This point will be discussed in § 4.

2.3. System initially prepared in an intermediate state between super-radiance and complete inversion

The case $M = 13$, $r = 37$ was studied using the spontaneous emission equation (5). Similarly to the super-radiant curves, figure 6 shows that the peak of the n-photon probability distribution shifts from $n = 0$ as the time progresses, although in a less dramatic way. The two curves of figure 6 correspond to

(a) $\tau = 0{\cdot}0165$ and (b) $\tau = 0{\cdot}033$.

3. Stimulated emission from N atoms

We have explored numerically two cases of stimulated emission using the general expression for the n-photon probability (equation (5)). In figure 7 the parameters

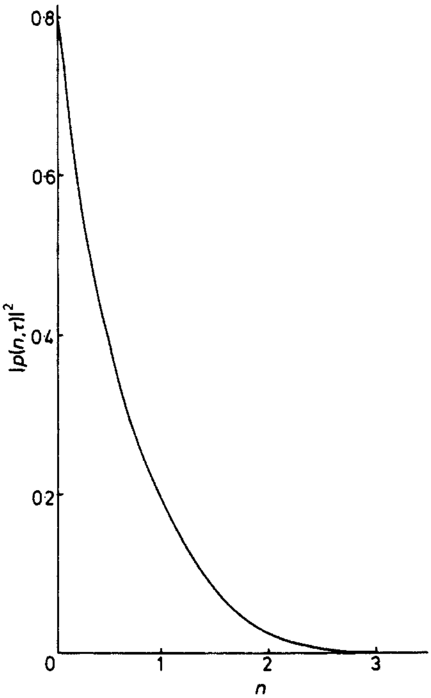

Figure 3. n-photon probability distribution versus the number of photons n for spontaneous emission ($n(0)=0$), when the atoms are initially prepared in the state of complete inversion ($M=r$).

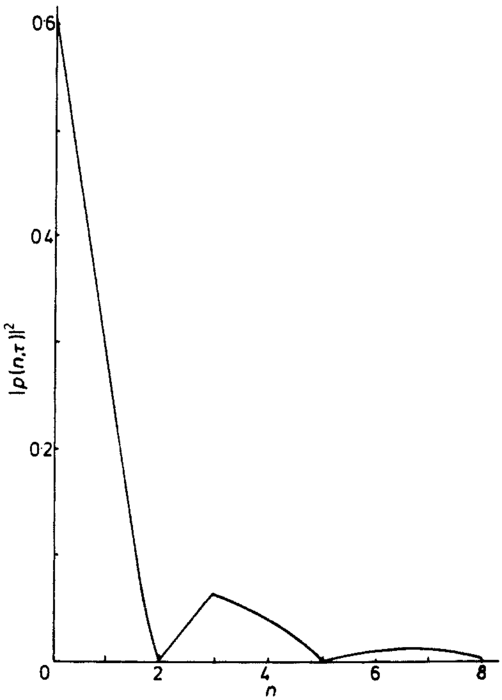

Figure 4. As figure 3 with $\tau = 0{\cdot}3$.

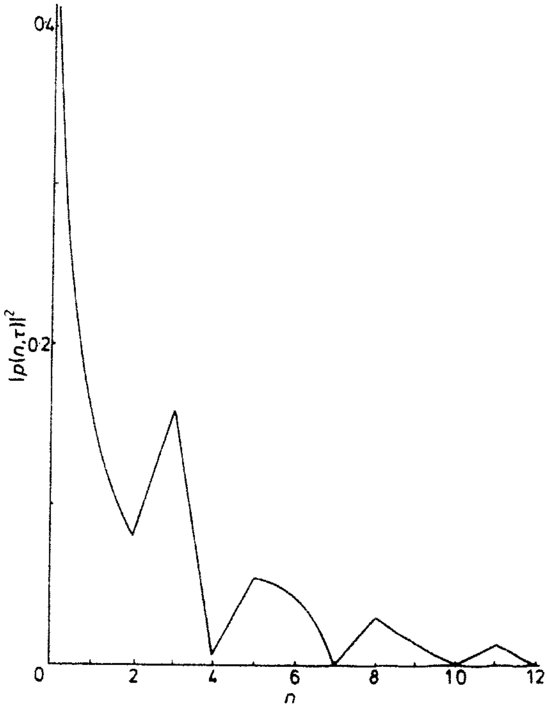

Figure 5. As figure 3 with $\tau = 0{\cdot}5$.

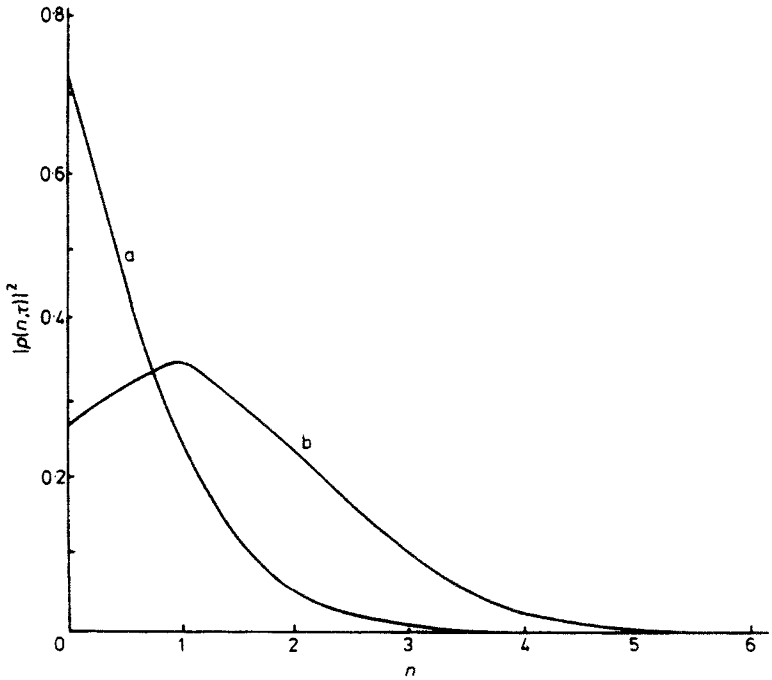

Figure 6. n-photon probability distribution versus the number of photons n for spontaneous emission ($n(0) = 0$), when the atoms are initially prepared in a state intermediate between the super-radiant and complete inversion states.

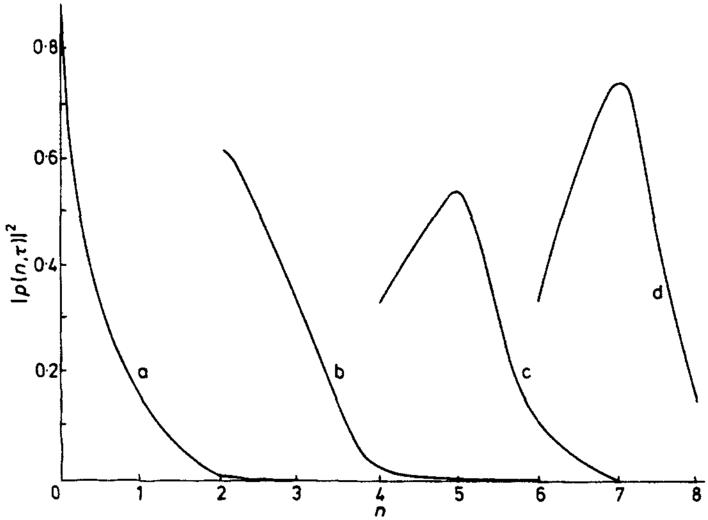

Figure 7. n-photon probability distribution versus the number of photons n for both stimulated and spontaneous emission.

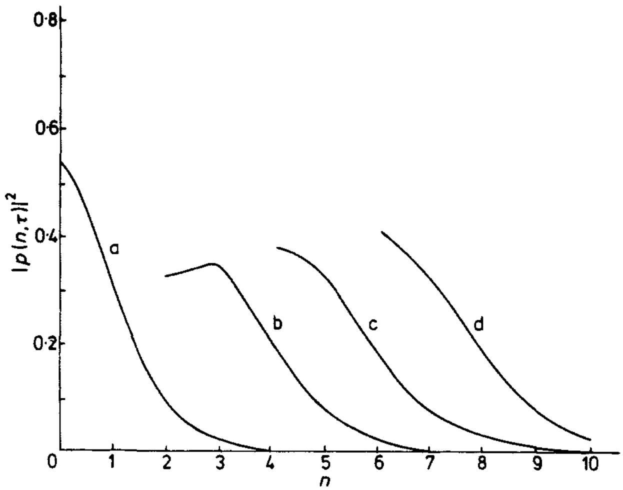

Figure 8. As figure 7 with $\tau = 0{\cdot}033$ and $n(0) = 0$ (a), 2 (b), 4 (c) and 6 (d).

$r = 25$, $M = 10$ and $\tau = 0{\cdot}0165$ were used for the cases (a) $n(0) = 0$, (b) $n(0) = 2$, (c) $n(0) = 4$ and (d) $n(0) = 6$. Curves (a) and (b) have no peaks and the maximum probability occurs for $n = n(0)$. Curves (c) and (d) peak at some $n > n(0)$. A similar result is obtained in figure 8, for $r = 25$ and $M = 10$ for the following cases: $n(0) = 0$ (curve (a)), $n(0) = 2$ (curve (b)), $n(0) = 4$ (curve (c)) and $n(0) = 6$ (curve (d)). τ was chosen to be $0{\cdot}033$.

4. Discussion

The numerical results for Dicke's model of N atoms interacting with radiation, in the restricted rotating-wave approximation, agree well with previous numerical and approximate solutions in all cases except that of complete inversion ($M = r$). In this case we have observed a consistent deviation of our results from the exact numerical results previously published. This is not surprising if one considers that the restricted rotating-wave approximation introduces a correction of a quantum nature into the model. This will therefore affect the case in which the quantum nature of light is important.

It is known that an analogy can be established between interacting atoms and a simple pendulum (Arecchi and Bonifacio 1965, McCall and Hahn 1967) in which a semi-classical approach is used. It is also known that when $M = r$ no radiation is emitted from the atoms semi-classically, which corresponds to the pendulum in unstable equilibrium. This case is therefore very sensitive to quantisation of the electromagnetic radiation, and one would expect a definite change when quantum corrections are introduced.

On the other hand, the interaction problem, when the atoms have been prepared in the super-radiant state, can be described well in the context of the semi-classical model, and it would thus be insensitive to quantum corrections.

Finally, it is worth mentioning that the procedure for obtaining the n-photon probability for the case of complete inversion can be extended to the super-radiant case, and a solution obtained for all times. This problem will be the subject of a future publication.

Acknowledgments

Many thanks are due to Mr Jaime Menendez, who generously wrote the difficult APL computer program used in this work. I would also like to thank Mr A Ronen for writing the FORTRAN program for the case $M = r$.

References

Arecchi F T and Bonifacio R 1965 *J. Quantum Electron.* **1** 163
Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A**6** 2211
Bonifacio R and Preparata G 1970 *Phys. Rev.* A**2** 336
Dicke R H 1954 *Phys. Rev.* **93** 99
Jaynes E T and Cummings F W 1963 *Proc. I.E.E.E.* **51** 89
McCall S L and Hahn E L 1967 *Phys. Rev. Lett.* **18** 908
Orszag M 1979 *J. Phys. A: Math. Gen.* **12** 2205
Walls D F and Barakat R 1970 *Phys. Rev.* A**1** 446